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THE ERROR OF POLYTOPAL APPROXIMATION WITH RESPECT TO THE SYMMETRIC DIFFERENCE METRIC AND THE L_p METRIC

BY

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ABSTRACT

Let M be a convex body in $\mathbb{R}^d$ with C^3_+ boundary. Polytopal approximation of M with respect to the symmetric difference metric (or the L_p metric) is considered, if the approximating polytope has at most n facets (or at most n vertices). The asymptotic behavior of the distance of the best approximating polytope is well-known; it is of order $n^{\frac{-2}{d-1}}$. This paper provides an estimate of order $n^{\frac{-2}{d-1}+\frac{-1}{8d^2}}$ for the error term.

1. Introduction

Assume that M is a convex body with C^2_+ boundary, namely, the boundary is C^2 and the Gauß–Kronecker curvature is positive everywhere. Let δ be either the symmetric difference metric or the L_p metric (see below for definition). Consider the polytope P_n (or $P_{(n)}$) with at most n vertices (at most n facets) minimizing $\delta(M, P_n)$ (or $\delta(M, P_{(n)})$). Since the middle of the century, much effort and many brilliant ideas have been put into obtaining asymptotic formulae for $\delta(M, P_n)$ and $\delta(M, P_{(n)})$ as n tends to infinity. The investigations were started by László Fejes Tóth (in dimensions 2 and 3, cf. [6]), and continued by McClure and Vitale in the plane, cf. [17]. The higher dimensional analogues needed new ideas. It was P. M. Gruber who first obtained asymptotic formulae (after the breakthrough

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by R. Schneider for the Hausdorff metric, which is easier to handle). By now, a whole theory has been developed by P. M. Gruber, R. Schneider, S. Glasauer and M. Ludwig (see the comprehensive surveys [11] and [12] for the history and for the state of art of this field).

The central problem of this paper is to estimate the error of the asymptotic formulae as n tends to infnity.

Note that the support function of a convex body K is defined as $h_K(u) = \max_{x\in K}\langle x,u\rangle$. Let M and P be convex bodies.

SYMMETRIC DIFFERENCE METRIC: $\delta_S(M,P)$ is the volume of the symmetric difference of M and P.

L_p METRIC, $p \geq 1$: $\delta_p(M,P) = \left(\int_{S^{d-1}} |h_M(u) - h_P(u)|^p\, du\right)^{\frac{1}{p}}$.

If $P \subset M$ then $\delta_1(M,P)$ is actually proportional to the deviation of the mean width, while minimizing $\delta_S(M,P)$ is equivalent to maximizing $V(P)$. On the other hand, the L_2 metric is a useful tool for stability of geometric inequalities (see H. Groemer [8]).

Assume that the boundary ∂M of the convex body M is C^2 (for definition and related notions, see R. Schneider [18], or the beginning of Section 3). Denote the second fundamental form at $x \in \partial M$ by Q_x, and hence the Gauß–Kronecker curvature is $\kappa(x) = \det Q_x$. Naturally, both Q_x and $\kappa(x)$ are continuous functions of $x \in \partial M$. The convexity of M yields that Q_x ($\kappa(x)$) is positive semidefinite (non-negative).

We say that ∂M is C^2_+ if Q_x is positive definite at each $x \in \partial M$; or equivalently, $\kappa(x)$ is positive at each $x \in \partial M$. The basic reference about the properties of smooth convex bodies is R. Schneider [18].

The paper deals with a convex body M such that Q_x is a Lipschitz function of $x \in \partial M$. Observe that this property is guaranteed if ∂M is C^3_+.

First we consider the symmetric difference metric. If P_n ($P_{(n)}$) is the polytope with at most n vertices (facets) minimizing $\delta_S(M,P_n)$ ($\delta_S(M,P_{(n)})$) then (see M. Ludwig [15])

$$\text{(1)} \qquad \delta_S(M,P_n) \sim \frac{1}{2}\,\mathrm{ldel}_{d-1}\cdot\left(\int_{\partial M}\kappa(x)^{\frac{1}{d+1}}\,dx\right)^{\frac{d+1}{d-1}}\cdot\frac{1}{n^{\frac{2}{d-1}}},$$

$$\text{(2)} \qquad \delta_S(M,P_{(n)}) \sim \frac{1}{2}\,\mathrm{ldiv}_{d-1}\cdot\left(\int_{\partial M}\kappa(x)^{\frac{1}{d+1}}\,dx\right)^{\frac{d+1}{d-1}}\cdot\frac{1}{n^{\frac{2}{d-1}}}.$$

Here ldel_{d-1} and ldiv_{d-1} are constants defined in $\mathbb{R}^{d-1}$, and independent of M and n. The expression $\int_{\partial M}\kappa(x)^{\frac{1}{d+1}}\,dx$ is the so-called affine surface area of M, and it is invariant under volume preserving affine transformations (see W. Blaschke [2], E. Lutwak [16] or K. Leichtweiß [13]).

Formulae analogous to (1) and (2) were proved earlier assuming that P_n is inscribed (cf. P. M. Gruber [9]), or that $P_{(n)}$ is circumscribed (cf. P. M. Gruber [10]). Here we also consider the cases if P_n is circumscribed, or $P_{(n)}$ is inscribed.

Our aim is to estimate the order of the error term. By $O(\cdot)$ we mean the Landau symbol where the implied constant depends on M.

THEOREM 1: *Assume that ∂M is C^2_+ and the second fundamental form Q_x is a Lipschitz function of x. If P_n ($P_{(n)}$) is the polytope with at most n vertices (facets) minimizing $\delta_S(M,P_n)$ ($\delta_S(M,P_{(n)})$) then*

$$\delta_S(M,P_n) = \left(1+O\left(n^{\frac{-1}{8d^2}}\right)\right)\cdot\frac{1}{2}\,\mathrm{ldel}_{d-1}\cdot\left(\int_{\partial M}\kappa(x)^{\frac{1}{d+1}}\,dx\right)^{\frac{d+1}{d-1}}\cdot\frac{1}{n^{\frac{2}{d-1}}},$$

$$\delta_S(M,P_{(n)}) = \left(1+O\left(n^{\frac{-1}{8d^2}}\right)\right)\cdot\frac{1}{2}\,\mathrm{ldiv}_{d-1}\cdot\left(\int_{\partial M}\kappa(x)^{\frac{1}{d+1}}\,dx\right)^{\frac{d+1}{d-1}}\cdot\frac{1}{n^{\frac{2}{d-1}}}.$$

One may impose the additional condition that the polytope is inscribed or circumscribed.

In the planar case, M. Ludwig managed to verify the formula

$$\delta_S(M,P_n) = a_2(M)\cdot\frac{1}{n^2} + a_4(M)\cdot\frac{1}{n^4} + O\left(\frac{1}{n^5}\right)$$

where $a_2(M)$ is given above, and $a_4(M)$ is a certain affine invariant function of M (see [14]). This points to the conjecture of P. M. Gruber (see [12]) that if ∂M is sufficiently smooth in $\mathbb{R}^d$ then there exists a series expansion of $\delta_S(M,P_n)$ (and probably also for the other metrics used in polytopal approximation).

Next we turn to the L_1 metric. It turns out that the problem is closely related to approximation by δ_S, as an argument of S. Glasauer shows (see S. Glasauer and P. M. Gruber [7], or Section 4).

THEOREM 2: *Assume that ∂M is C^2_+ and the second fundamental form Q_x is a Lipschitz function of x. If P_n ($P_{(n)}$) is the polytope with at most n vertices (facets) minimizing $\delta_1(M,P_n)$ ($\delta_1(M,P_{(n)})$) then*

$$\delta_1(M,P_n) = \left(1+O\left(n^{\frac{-1}{8d^2}}\right)\right)\cdot\frac{1}{2}\,\mathrm{ldiv}_{d-1}\cdot\left(\int_{\partial M}\kappa(x)^{\frac{d}{d+1}}\,dx\right)^{\frac{d+1}{d-1}}\cdot\frac{1}{n^{\frac{2}{d-1}}},$$

$$\delta_1(M,P_{(n)}) = \left(1+O\left(n^{\frac{-1}{8d^2}}\right)\right)\cdot\frac{1}{2}\,\mathrm{ldel}_{d-1}\cdot\left(\int_{\partial M}\kappa(x)^{\frac{d}{d+1}}\,dx\right)^{\frac{d+1}{d-1}}\cdot\frac{1}{n^{\frac{2}{d-1}}}.$$

One may impose the additional condition that the polytope is inscribed or circumscribed.

Observe that the role of ldiv and ldel has been interchanged. This not an accident, since the correspondence between δ_1 and δ_S is via polarity (see Section 4). In particular, the constants in the asymptotic formulae for δ_S and for δ_1 correspond to each other; namely, the role of vertices and facets, and the words inscribed and circumscribed should be interchanged in the statements.

If Q_x is assumed to be only continuous then the corresponding asymptotic formulae were proved in S. Glasauer and P. M. Gruber [7] and M. Ludwig [15].

Finally, we consider the L_p metric, $p > 1$, only if the polytope is inscribed, and the number of vertices is bounded. Let ∂M be C^2_+, and for $p > 1$, assume that $P_n \subset M$ is the polytope with n vertices minimizing $\delta_p(M, P_n)$. Then we prove (following the method of S. Glasauer and P. M. Gruber [7]) that

$$\delta_p(M, P_n) \sim \frac{1}{2}\,\mathrm{div}_{d-1} \cdot \left(\int_{\partial M} \kappa(x)^{\frac{d-1+p}{d-1+2p}}\,dx\right)^{\frac{d-1+2p}{p(d-1)}} \cdot \frac{1}{n^{\frac{2}{d-1}}}. \tag{3}$$

We strengthen this formula if the second fundamental form is Lipschitz:

THEOREM 3: *Assume that ∂M is C^2_+ and the second fundamental form Q_x is a Lipschitz function of x. If $p > 1$ and $P_n \subset M$ is the polytope with at most n vertices minimizing $\delta_p(M, P_n)$ then*

$$\delta_p(M, P_n) = \left(1 + O\left(n^{\frac{-p}{20d(d+p)}}\right)\right) \cdot c_{p,d} \cdot \left(\int_{\partial M} \kappa(x)^{\frac{d-1+p}{d-1+2p}}\,dx\right)^{\frac{d-1+2p}{p(d-1)}} \cdot \frac{1}{n^{\frac{2}{d-1}}}.$$

Observe that the error term gets better if p increases. In particular, if say $p \geq d$ then the exponent in the error term is of order $1/d$. This was proved in the case of Hausdorff metric ($p = \infty$) in K. Böröczky, Jr. [4].

In order to get any error term, it seems to be essential that the fundamental form is Lipschitz and positive definite. On the other hand, the asymptotic formulae can be proved even allowing the curvature to be zero (see K. Böröczky, Jr. [3]).

The paper is organized as follows: First we prove Theorem 1 in Sections 2 and 3. The two theorems about the L_p metrics are verified in Section 4, relying heavily on Theorem 1 and its proof.

We close the paper showing that the analogous formulae hold for general convex hypersurfaces (see Corollary 2) with the appropriate definition of distances and approximating polytopal hypersurfaces.

Some words about notation: We write $f \ll g$ or $f = O(g)$ if there exists a constant $c > 0$ depending on M such that $|f| < c \cdot g$. If $f \ll g$ and $f \gg g$ then write $f \approx g$. Note the constants in Sections 2.1, 2.2 and 2.3 depend only on d.

In $\mathbb{R}^d$, we fix a scalar product $\langle \cdot, \cdot, \rangle$, and denote by $\| \cdot \|$ the corresponding norm. The induced quadratic form defined in $\mathbb{R}^{d-1}$ is denoted by q_0. When speaking about inradius, *etc.* of bodies in $\mathbb{R}^{d-1}$, we mean the metric determined by q_0.

On the other hand, we frequently use some other positive definite quadratic form q. If we use the metric defined by q then we say that inradius, it etc. is defined with respect to q.

2. Symmetric difference metric: power diagrams

Let M be a convex body with C^2_+ boundary.

We present the proof only for the case if the number of vertices is given. The arguments can be easily transferred to the case if the number of facets is bounded, and the necessary changes are explained at the end of the section.

Note that when piecing patches, we take the convex hull of the patches if the number of vertices is bounded, and intersection of convex shapes corresponding to the patches if the number of facets is bounded. If M is a convex body and P and Q are polytopes then

$$\delta_S(M, P \cap Q) < \delta_S(M, P) + \delta_S(M, Q).$$

On the other hand, it is much harder to control $\delta_S(M, \mathrm{conv}(P, Q))$ in terms of $\delta_S(M, P)$ and $\delta_S(M, Q)$. Therefore the case where the number of vertices is bounded is more complicated. The only exception is if the polytopes are supposed to be inscribed. Then one does not have to worry about piecing patches, since the larger the polytope the smaller the distance.

The asymptotic formula (1) encodes two statements: First, that for any $\varepsilon > 0$ and large n, the best approximating polytope P_n satisfies

$$\delta_S(M, P_n) > (1 - \varepsilon) \cdot \frac{1}{2} \,\mathrm{ldel}_{d-1} \cdot \left(\int_{\partial M} \kappa(x)^{\frac{1}{d+1}} \, dx \right)^{\frac{d+1}{d-1}} \cdot \frac{1}{n^{\frac{2}{d-1}}}.$$

This estimate we call lower bound. On the other hand, for any large n we construct a polytope Q_n with at most n vertices such that

$$\delta_S(M, Q_n) < (1 + \varepsilon) \cdot \frac{1}{2} \,\mathrm{ldel}_{d-1} \cdot \left(\int_{\partial M} \kappa(x)^{\frac{1}{d+1}} \, dx \right)^{\frac{d+1}{d-1}} \cdot \frac{1}{n^{\frac{2}{d-1}}}.$$

This estimate we call upper bound.

In case of Theorem 1, ε will be of the form $O\left(n^{\frac{-1}{8d^2}}\right)$.

The proof of (1) and Theorem 1 breaks down into two main steps: First we consider polytopal approximation of paraboloids, which in turn leads to power diagrams in $\mathbb{R}^{d-1}$. Then subdivide ∂M into almost paraboloid patches, transfer the results about paraboloids to these patches, and piece the estimates for the patches into a global estimate.

2.1 Power diagrams.

Let q be a positive definite quadratic form. Power diagrams are discussed say in [1] from an algorithmic prospective. The term Laguerre tiling is also used (this is the explanation for the "l" in ldel and ldiv). Actually, "del" stands in honour of Delone, and "div" stands in honour Dirichlet, in conjunction with the tilings named after them.

For us, a **power diagram** $T = \{\Pi_i, a_i, r_i\}$ with respect to q is a finite family of convex, compact $(d-1)$-polytopes $\{\Pi_i\}$, centers $\{a_i\}$ and real numbers $\{r_i\}$ such that the interiors of the Π_i's are disjoint, and $q(z-a_i) - r_i \le q(z-a_j) - r_j$ holds if $z \in \Pi_i$. Here the Π_i's are the facets of T, and in general, the union of the k-faces is the set of k-faces of T. We say that T covers a set C if the union of the facets in T covers C.

Call T circumscribed if $r_i \le 0$, and inscribed if $q(z-a_i) \le r_i$ holds for $z \in \Pi_i$.

Denote by T^+ the union of all Π_i and the ellipsoids $q(z-a_i) \le r_i$, and define

$$\Omega_i = \left\{ z \in T^+ \colon q(z-a_i) - r_i \le q(z-a_j) - r_j \text{ for each } j \right\}.$$

The following observation connects power diagrams to polytopal approximation: Let Y be the union of a subfamily $\{F_i\}$ of facets of a polytope P in $\mathbb{R}^d$ so that the exterior normals point downwards (we assume that $\mathbb{R}^{d-1}$ is embedded into $\mathbb{R}^d$). Define a_i so that the tangent hyperplane at $(a_i, q(a_i))$ to the graph of q is parallel to affF_i. Then there exists some r_i so that affF_i is the graph of $q(a_i) + l_{a_i}(z-a_i) + r_i$ where l_{a_i} is the derivative of q at a_i. Denote by Π_i the orthogonal projection of F_i into $\mathbb{R}^{d-1}$, and assume that Y is the graph of the piecewise linear function φ_i. Since $q(z) = q(a_i) + l_{a_i}(z-a_i) + q(z-a_i)$, we deduce that

$$f(z) - \varphi(z) = q(z-a_i) - r_i.$$

In particular, $T = \{\Pi_i, a_i, r_i\}$ is a power diagram with respect to q. For example, the ellipsoids $q(z-a_i) \le r_i$ are the projections of the caps cut off by affF_i from the graph of q.

This process can be reversed, and we denote by φ_T the piecewise linear function on T^+ associated to T.

Observe that T is circumscribed (inscribed) if and only if Y is below (above) the graph of q.

Finally, define

$$v(T)=\sum_i \int_{\Omega_i} |q(z-a_i)-r_i|\,dz,$$

and for any convex body C in $\mathbb{R}^{d-1}$ set

$$\begin{aligned} v_n(C,q) &= \inf_{T \text{covers} J} \{v(T)\text{: the number of vertices in } C \text{ is at most } n\}, \\ v_{(n)}(C,q) &= \inf_{T \text{covers} J} \{v(T)\text{: the number of tiles is at most } n\}. \end{aligned}$$

For $\lambda>0$, denote by λT the power diagram $\{\lambda\Pi_i, \lambda a_i, \lambda^2 r_i\}$ with respect to the same quadratic form q. Observe that

$$v(\lambda T)=\lambda^{d+1}\cdot v(T).$$

Denote the unit ball in $\mathbb{R}^{d-1}$ by B, and the $(d-1)$-dimensional Lebesgue measure by $|\cdot|$.

For a convex body C in $\mathbb{R}^{d-1}$, let $\varrho(C)$ be the inradius of C. Now if $t\le\varrho(C)$ then

$$|\partial C + t\,B| \ll \frac{|C|}{\varrho(C)}\cdot t. \tag{4}$$

The next estimate is contained implicitly in M. Ludwig [15], but we prefer to give a quick proof for the sake of completeness.

PROPOSITION 2.1: *Let q be a positive definite quadratic form, and denote by ϱ the inradius of C with respect to q. If $n\cdot\varrho^{d-1}>|C|$ then*

$$v_n(C,q)\approx(\det q)^{\frac{1}{d-1}}\cdot|C|^{\frac{d+1}{d-1}}\cdot\frac{1}{n^{\frac{2}{d-1}}}.$$

Proof: We may assume that $q=q_0$. Then the upper bound follows easily by taking say power diagrams constructed with the help of a square grid. The condition on n ensures that the boundary does not cause problems (see (4)).

Turning to the lower bound, let the power diagram $T=\{\Pi_i,a_i,r_i\}$ cover C with having at most n vertices in C. We may assume that each tile is a simplex. For each vertex u, define $St(u)$ as the union of the tiles of T containing u. Observe that $St(u)$ is starshaped with respect to u, namely, $\operatorname{conv}\{u,y\}$ is a subset of S for any $y\in S$. In addition, the intersection of any ray starting from u and $St(u)$ is contained in one of the tiles.

For the time being, assume that u is the origin. If $z \in \Pi_i$ then set $a(z) = a_i$ and $r(z) = r_i$ so that, along any open ray starting from u, $a(z)$ and $r(z)$ are constants. Denote by $R(v)$, $v \in \mathbb{R}^{d-2}$, the radial function of $St(u)$.

Note that

$$\min_{a,r\in\mathbb{R}} \int_0^1 \left|(t-a)^2 - r\right| t^{d-2}\, dt > \frac{1}{e} \cdot \min_{a,r\in\mathbb{R}} \int_{1-\frac{1}{d}}^1 \left|(t-a)^2 - r\right|\, dt \gg 1.$$

Therefore using polar coordinates and Hölder's inequality yields that

$$\int_{St(u)} \left|(z-a(z))^2 - r(z)\right|\, dz \gg \int_{S^{d-2}} R(u)^{d+1} du \gg \left(\int_{S^{d-2}} R(u)^{d-1} du\right)^{\frac{d+1}{d-1}}.$$

The last expression is $\approx |St(u)|^{\frac{d+1}{d-1}}$. Since each tile is a simplex, we deduce that

$$d \cdot v(T) \gg \sum_{u\in C \text{ vertex}} |St(u)|^{\frac{d+1}{d-1}}.$$

On the other hand, we have at most n starshaped set $St(u)$ and

$$\sum_{u\in C \text{ vertex}} |St(u)| > |C|.$$

Therefore Jensen's inequality yields the proposition. ∎

2.2 Some auxiliary statements.

First we show that a hypersurface well approximating ∂M lies really close to ∂M:

Proposition 2.2: *Assume that on an open subset of $\mathbb{R}^{d-1}$, q is a positive definite quadratic form and f is a C^2 function such that the quadratic form q_z representing the second derivative of f at z satisfies $q \le \frac{1}{2}q_z \le 2q$ for any z. If φ is a convex function and $|f(a) - \varphi(a)| \ge R$ for some $R > 0$ then*

$$\int_{q(z-a)\le R} |f(z) - \varphi(z)|\, dz \gg (\det q)^{-\frac{1}{2}} \cdot R^{\frac{d+1}{2}}.$$

Proof: We may assume that $q = q_0$. Subtracting the equation of the tangent hyperplane at $(a, f(a))$ yields that we may also assume that the derivative of f is zero at a. Using polar coordinates around a in $\mathbb{R}^{d-1}$ reduces the problem to the following one: If f and φ are convex, increasing functions on $[0, \sqrt{R}]$ such that $f(0) = 0$, $|\varphi(0)| \ge R$ and $1 \le \frac{1}{2}f''(t) \le 2$ then

$$\int_0^{\sqrt{R}} |f(t) - \varphi(t)| t^{d-2}\, dt \gg R^{\frac{d+1}{2}}.$$

Here we may assume that $\varphi(0) = -R$. If φ minimizes the integral above, then for some $0 < t_0 < t_1$, φ is linear on $[0, t_1]$, $f(t_i) = \varphi(t_i)$, and

$$\int_0^{t_0} t^{d-1}\,dt = \int_{t_0}^{t_1} t^{d-1}\,dt.$$

Now $t_1 \gg \sqrt{R}$ yields that $t_0 = 2^{-1/d} \cdot t_1 \gg \sqrt{R}$. We deduce that $\varphi(t) \leq 0$ if $t \leq c\sqrt{R}$ holds for some c depending on d, which in turn yields the Proposition. ∎

For piecing lower bounds, we use the following consequence of Hölder's inequality: Assume that μ_i, n_i, $i = 1, \ldots, k$, are positive numbers. Then

$$\mu_i^{\frac{d+1}{d-1}} \cdot \frac{1}{n_i^{\frac{2}{d-1}}} \geq \left(\sum_i \mu_i\right)^{\frac{d+1}{d-1}} \cdot \frac{1}{(\sum_i n_i)^{\frac{2}{d-1}}}. \tag{5}$$

On the other hand, when piecing polytopal patches, we need the following statement (see the remarks about piecing at the beginning of Section 2):

Proposition 2.3: *Let $\{\varphi_i\}$ and φ form a finite family of convex functions in $\mathbb{R}^{d-1}$ such that the domain $\tilde{\sigma}_i$ (σ) of φ_i (of φ) is a simplicial complex and φ_i (φ) is linear on each simplex. Assume that there exist subcomplices σ_i of $\tilde{\sigma}_i$ such that the σ_i's cover σ, the graph of φ is the lower convex envelope of the graphs of the $\varphi_i\,|_{\sigma_i}$, and if the simplex Π of σ intersects σ_i then Π is covered by $\tilde{\sigma}_i$.*

Now let q be a positive definite quadratic form. Assume that f is a C^2 function on the union of the $\tilde{\sigma}_i$'s such that the quadratic form q_z representing the second derivative of f at z satisfies $q \leq \frac{1}{2}q_z \leq 2q$ for any z. Then

$$int_\sigma |f(z) - \varphi(z)|\,dz \ll \sum_i \int_{\tilde{\sigma}_i} |f(z) - \varphi_i(z)|\,dz.$$

Proof: It is sufficient to prove that for any simplex Π in σ, the inequality

$$\int_\Pi |f(z) - \varphi(z)|\,dz \ll \sum_i \int_{\Pi\cap\tilde{\sigma}_i} |f(z) - \varphi_i(z)|\,dz \tag{6}$$

holds.

Let Π be a simplex of σ. Define $v \in \Pi$ by

$$|f(v) - \varphi_i(v)| = \max_{z\in\Pi} \{|f(z) - \varphi(z)|\}. \tag{7}$$

First assume that $f(v) < \varphi(v)$, and let Π^* be the set of $z \in \Pi$ such that $f(z) \leq \varphi(z)$. Then (7) yields that

$$\int_\Pi |f(z) - \varphi(z)|\,dz \ll \int_{\Pi^*} \varphi(z) - f(z)\,dz,$$

and hence (6) follows as $\varphi \le \varphi_i$ for each i.

Next assume that $f(v) > \varphi(v)$. Then v is a vertex of Π, and there exists a φ_i such that $\varphi(v) = \varphi_i(v)$. We prove that

$$\int_\Pi |f(z) - \varphi(z)|\, dz \ll \int_\Pi |f(z) - \varphi_i(z)|\, dz. \tag{8}$$

Adding a linear function to each function, and using polar coordianates, the problem can be translated in the following one dimensional one: Assume that $1 \le f''(t) \le 2$ on $[a,b]$, φ is a constant and φ_i is an increasing convex function. If $\varphi_i(a) = \varphi < f(a)$, and

$$\max_{[a,b]} |f(t) - \varphi(t)| = f(a) - \varphi(a),$$

then

$$\int_a^b |f(t) - \varphi|\, t^{d-2} dt \ll \int_a^b |f(t) - \varphi_i(t)|\, t^{d-2} dt.$$

We may assume that $f(t) \ge \varphi$ for $t \in [a,b]$. Set $c = \max_{t\in[a,b]}\{\varphi_i(t) \le f(t)\}$. If $c \le (a+b)/2$ then

$$\int_a^b |f(t) - \varphi|\, t^{d-2} dt \ll \int_c^b |f(t) - \varphi_i(t)|\, t^{d-2} dt,$$

and if $c \ge (a+b)/2$ then

$$\int_a^b |f(t) - \varphi|\, t^{d-2} dt \ll \int_a^c |f(t) - \varphi_i(t)|\, t^{d-2} dt.$$

With this, the proof of the proposition is complete. ∎

2.3 Covering a parallelotope.

Let W be the unit cube $W = [-\frac{1}{2}, \frac{1}{2}]^{d-1}$ in $\mathbb{R}^{d-1}$. Assume that T is a power diagram covering W such that $v(T) < 2\cdot v_n(W, q_0)$. We deduce by Proposition 2.1 and Proposition 2.2 that there exists a positive α depending on d such that for any $z \in T^+$, we have

$$|q_0(z) - \varphi_T(z)| < \alpha \cdot n^{\frac{-4}{(d-1)(d+1)}}. \tag{9}$$

On the other hand, there exists a $\gamma > 0$ depending on α such that if (9) holds then, for any Ω_i associated to a tile Π_i of T, we have

$$\operatorname{diam} \Omega_i < \gamma \cdot n^{\frac{-2}{(d-1)(d+1)}}. \tag{10}$$

First we consider a general approximation. The error term in our formulae is usually of the form $O\big(n^{\frac{-1}{\alpha d^2}}\big)$ for certain α. We do not make an attempt to find the optimal α.

PROPOSITION 2.4:

$$v_n(W, q_0) = \left(1 + O\left(n^{\frac{-1}{3d^2}}\right)\right) \cdot \mathrm{ldel}_{d-1} \cdot \frac{1}{n^{\frac{2}{d-1}}}$$

for the suitable ldel_{d-1} *depending only on d.*

Proof: Denote by $\widetilde{T}_n$ a power diagram covering W, having at most n vertices in W and satisfying $v(\widetilde{T}_n) = v_n(W, q_0)$. Define $c_n = v_n(W, q_0) \cdot n^{\frac{2}{d-1}}$.

First we claim that if $m > \log k$ and $m^{d-1}k \leq N < (m+1)^{d-1}k$ then

$$c_N < \left(1 + O\left(\frac{1}{m}\right) + O\left(\frac{1}{k^{\frac{2}{(d-1)(d+1)}}}\right)\right) \cdot c_k. \tag{11}$$

Here the condition $m > \log k$ ensures that m tends to infinity with k.

Set

$$\varepsilon = k^{\frac{-2}{(d-1)(d+1)}},$$

and denote by $T_k = \{\Pi_i, a_i, r_i\}_{i=1,\dots,t}$ the power diagram, which consists of the tiles of $(1+2\gamma\cdot\varepsilon)\widetilde{T}_k$ intersecting W. Then $v(T_k) \leq (1+O(\varepsilon))v_k(W, q_0)$, T_k covers W and the number of all the vertices of T_k is at most k (independent of whether they are in W).

Denote $T_k^+ \backslash (1 - 4\gamma \cdot \varepsilon)W$ by T_k^-, which is contained in $(1 + 4\gamma \cdot \varepsilon)W$ if k is large. In addition, the minimality of $v(\widetilde{T}_k)$ yields that

$$\sum_{i=1}^{t} \int_{\Omega_i \cap T_k^-} |(z - a_i)^2 - r_i| \, dz \ll \varepsilon \cdot v(\widetilde{T}_k). \tag{12}$$

Let $y_1, \dots, y_{m^d}$ be the vectors such that $y_1 + \frac{1}{m}W, \dots, y_{m^d} + \frac{1}{m}W$ tiles W. For $j = 1, \dots, m^d$, denote by φ_j the piecewise linear function associated to the power diagram $y_j + \frac{1}{m}T_k$. Consider the convex hull of the union of the vertices of the graph of each φ_j. Now there exists a power diagram T_N such that the graph of φ_{T_N} over the tiles of T_N is the lower envelope of the convex hull.

Readily T_N covers W, and it has at most $m^d k \leq N$ vertices. For any $z \in T_N^+$, we have

$$|q_0(z) - \varphi_{T_N}(z)| < \alpha \cdot (m^d k)^{\frac{-4}{(d-1)(d+1)}}$$

because this property holds for each φ_j. Therefore (10) yields that if a vertex of a tile Π is coming from $y_j + \frac{1}{m}T_k$ then Π is covered by the tiles of

$$y_j + \frac{1}{m} \cdot (1 + 2\gamma \cdot \varepsilon) \cdot \widetilde{T}_k.$$

In particular, Proposition 2.3 can be applied, and (12) yields that

$$v(T_N) < (1 + O(\varepsilon)) \cdot c_k \cdot (m^d k)^{\frac{-2}{d-1}}.$$

In turn, we conclude (11).

We deduce by Proposition 2.1 that the sequence $\{c_n\}$ is bounded from below and above by positive numbers. Thus (11) yields that $\mathrm{ldel}_{d-1} = \lim_{n\to\infty} c_n$ exists and positive. In addition,

$$(13) \qquad c_n > \left(1 - O\left(n^{\frac{-2}{(d-1)(d+1)}}\right)\right) \cdot \mathrm{ldel}_{d-1}.$$

Next we prove that for large k, there exists k_0 with

$$(14) \qquad k^{1-\frac{1}{2d}} < k_0 < k^{1+\frac{1}{d^2}}$$

such that

$$(15) \qquad c_N > \left(1 - O\left(k^{\frac{-1}{2d^2}}\right)\right) \cdot c_{k_0}$$

holds for $N = k^{1+\frac{1}{d-1}}$, and k_0 tends to infinity with k.

Set $m = k^{\frac{1}{(d-1)^2}}$, which satisfies $N = m^{d-1}k$. Consider the tiling $\{y_j + \frac{1}{m}W\}$ of W by the m^{d-1} translates of $\frac{1}{m}W$. Observe that if Π is a facet of $\widetilde{T}_N$ then

$$\mathrm{diam}\Pi < \gamma \cdot N^{\frac{-2}{(d-1)(d+1)}} = \frac{\gamma}{m^{\frac{d-1}{d+1}}} \cdot \frac{1}{m}.$$

Denote by N_j the number of facets of $\widetilde{T}_N$ intersecting

$$\left\{W_j = y_j + \left(1 - \frac{\gamma}{m^{\frac{d-1}{d+1}}}\right) \cdot \frac{1}{m} W\right\}$$

where no facet of $\widetilde{T}_N$ meets two of the W_j's.

Call an N_j all right if $k^{1-\frac{1}{2d}} \le N_j \le k^{1+\frac{1}{d^2}}$. We claim that there exists an all right N_j satisfying $c_{N_j} > \left(1 + k^{\frac{-1}{2d^2}}\right) c_N$.

Suppose to the contrary that such an N_j does not exist. Observe that the number of large N_j's with $N_j > k^{1+\frac{1}{d^2}}$ is readily at most $m^{d-1}k^{\frac{-1}{d^2}}$. On the other hand, Proposition 2.1 applied to W and the W_j's yields that the number of N_j's with $N_j \le k^{1-\frac{1}{2d}}$ is also $O\left(m^{d-1}k^{\frac{-1}{d^2}}\right)$. In particular, the volume covered by the W_j's corresponding to all right N_j's is $1 - O(k^{\frac{-1}{d^2}})$.

Now (5) and the indirect hypothesis yield for large k that

$$v\left(\widetilde{T}_N\right) > \left(1 + k^{\frac{-1}{2d^2}}\right)\left(1 - O\left(k^{\frac{-1}{d^2}}\right)\right) \cdot v\left(\widetilde{T}_N\right) > v\left(\widetilde{T}_N\right).$$

This is absurd, therefore the all right N_j with $c_{N_j} \leq \left(1 + k^{\frac{-1}{2d^2}}\right) c_N$ can be chosen as the k_0 in (15).

Finally, we claim that for large n, there exists $N(n) > n^{1+\frac{1}{3d}}$ satisfying

$$c_{N(n)} > \left(1 - O\left(n^{\frac{-1}{3d^2}}\right)\right) c_n. \tag{16}$$

In order to prove the claim, apply (15) with $k = n^{1-\frac{1}{2d}}$. We deduce that there exists $N(n) > n^{1+\frac{1}{3d}}$ such that

$$c_{N(n)} > \left(1 - O\left(n^{\frac{-1}{3d^2}}\right)\right) c_{k_0}.$$

On the other hand, (11) yields that

$$c_n < \left(1 + O\left(n^{\frac{-1}{3d^2}}\right)\right) c_{k_0}$$

because k_0 satisfies (14). In turn, we deduce the claim.

Now for any large n, construct a series $\{n_l\}$ with $n_0 = n$ and $n_{l+1} = N(n_l)$. Applying (16) to this series shows that $\mathrm{ldel}_{d-1} > \left(1 - O\left(n^{\frac{-1}{3d^2}}\right)\right) c_n$, and hence the proof of the proposition is finally complete. ∎

Now let P be a parallelotope such that the ratio of the circum- and the inradius is at most $2\sqrt{d}$. Then the same argument can be repeated about $v_n(P, q_0)$, using the family $\{y_i + \frac{1}{m}P\}$. In particular, we deduce

Proposition 2.5: *In $\mathbb{R}^{d-1}$, let P be a parallelotope and let q be a positive definite quadratic form. If the ratio of the circum- and the inradius with respect to q is at most $2\sqrt{d}$ then*

$$v_n(P, q) = \left(1 + O\left(n^{\frac{-1}{3d^2}}\right)\right) \cdot \mathrm{ldel}_{d-1} \cdot (\det q)^{\frac{1}{d-1}} \cdot |P|^{\frac{d+1}{d-1}} \cdot \frac{1}{n^{\frac{2}{d-1}}}.$$

Remark: The proof shows that the same estimate holds even assuming that the total number of vertices of the power diagram is at most n in Proposition 2.5, and each tile intersects P.

If the power diagram is supposed to be inscribed (circumscribed) then the analogue of Proposition 2.5 still holds, with the appropriate constant $c_{\text{inscribed}}$ ($c_{\text{circumscribed}}$). Note that $c_{\text{inscribed}}$ was baptized as del_{d-1} in P. M. Gruber [9].

3. Symmetric difference metric: approximating the boundary

3.1 SUBDIVIDING THE BOUNDARY.

We describe the properties of the convex body M with C^2_+ boundary we need in the sequel: Identify the tangent hyperplane at $x \in \partial M$ with $\mathbb{R}^{d-1}$. Then an open neighborhood U of x in ∂M is the graph of a convex C^2 function f defined in the projection V of U into $\mathbb{R}^{d-1}$.

Denote by l_z the derivative of f at z and by q_z the quadratic form representing the second derivative of f. Here $Q_x = q_x$, q_z is positive definite since ∂M is C^2_+, and actually q_z is a Lipschitz function of z if and only if the second fundamental form is Lipschitz on ∂M. Fix some $y \in V$. We deduce using the Taylor expansion of f that

$$f(z) = f(y) + l_y(z-y) + \frac{1}{2}q_w(z-y)$$

where $w = y + t(z-y)$ for some $0 < t < 1$. In particular, the Gauß–Kronecker curvature at $w = (z, f(z))$ can be expressed as

$$\kappa(w) = \frac{\det q_z}{(1+\|l_z\|^2)^{\frac{d+1}{2}}}.$$

LEMMA 1: *Assume that ∂M is C^2_+, Q_x is a Lipschitz function of x on ∂M, and consider the positive Lipschitz functions ψ (ϱ) in a neighborhood of ∂M in $\mathbb{R}^d$ (on ∂M).*

Then for large m, there exist pairs of open Jordan measurable subsets $\Sigma_\beta \subset \widetilde{\Sigma}_\beta$ of ∂M and hyperplanes H_β, quadratic forms q_β and constants ψ_β with the following properties: $\#\{\Sigma_\beta\} \approx m$;

(i) *either the distance between any two $\widetilde{\Sigma}_\beta$'s is $\gg m^{\frac{-2}{d-1}}$ and*

$$\sum_\beta \int_{\Sigma_\beta} \varrho(x)\,dx > \left(1 - O\left(\frac{1}{m^{\frac{1}{d-1}}}\right)\right) \cdot \int_{\partial M} \varrho(x)\,dx;$$

or any $\sigma \subset \partial M$ with $\operatorname{diam}\sigma \ll m^{\frac{-2}{d-1}}$ is contained in some $\{\widetilde{\Sigma}_\beta\}$ and

$$\sum_\beta \int_{\widetilde{\Sigma}_\beta} \varrho(x)\,dx < \left(1 + O\left(\frac{1}{m^{\frac{1}{d-1}}}\right)\right) \cdot \int_{\partial M} \varrho(x)\,dx.$$

In addition, call a Σ_β boring if there exists an $x \in \Sigma_\beta$ with distance at least $\gg m^{\frac{-2}{d-1}}$ from $\partial\Sigma_\beta$, and x is contained also in another Σ_β. Then

$$\sum_{\Sigma_\beta \text{ boring}} \int_{\widetilde{\Sigma}_\beta} \varrho(x)\,dx \ll \frac{1}{m^{\frac{1}{d-1}}} \cdot \int_{\partial M} \varrho(x)\,dx;$$

(ii) $\widetilde{\Sigma}_\beta$ (Σ_β) *is the graph of a* C^2 *function* f_β *on some* $\widetilde{\Phi}_\beta \subset H_\beta$ ($\Phi_\beta \subset H_\beta$);

(iii) Φ_β *is a parallelotope,* $|\Phi_\beta| \approx 1/m$, *the ratio of the circum- and inradius of* Φ_β *is at most* $2\sqrt{d}$, *and if* x *is in the boundary of* $\widetilde{\Phi}_\beta$ *then the distance of* x *and* Φ_β *is* $\approx m^{\frac{-2}{d-1}}$;

(iv) *if* l_z *is the derivative of* f_β *at* z *then* $\|l_z\| = O\Big(\frac{1}{m^{\frac{1}{d-1}}}\Big)$;

(v) *for the quadratic form* q_z *representing the second derivative of* f_β *at* z, *we have*

$$q_\beta \le \frac{1}{2} q_z \le \left(1 + O\left(m^{\frac{-1}{d-1}}\right)\right) \cdot q_\beta;$$

(vi) *if* z *is in a neighborhood of* Σ_β *in* $\mathbb{R}^d$ *then*

$$\psi_\beta < \psi(z) < \left(1 + O\left(m^{\frac{-1}{d-1}}\right)\right) \cdot \psi_\beta.$$

Proof: We use in the proof that there exists some $\omega > 0$ depending on M such that for any $x \in \partial M$ and any tangent vector u at x, we have

$$\omega \|u\| < Q_x(u) < \omega^{-1} \|u\|.$$

Denote by $\nu(x)$ the exterior unit normal at $x \in \partial M$. There exist a finite family $\{G_i\}$ of hyperplanes avoiding M and an open convex set Z_i with full dimension in each G_i such that the following holds: Denote by X_i the points of ∂M on the side of G_i whose orthogonal projection into G_i lands in X_i, and let $x_i \in X_i$ be the point such that $\nu(x_i)$ is normal to G_i. Then $\{X_i\}$ cover ∂M. In addition, $|\langle \nu(x), \nu(x_i)\rangle| > 0.99$ and $0.99 Q_{x_i} \le Q_x \le 1.01 Q_{x_i}$ hold if $x \in X_i$.

Now for large m, consider grids in each G_i such that the grid in G_i has a fundamental parallelotope W_i which is a $(d-1)$-cube with respect to Q_{x_i} and $|W_i| = \frac{1}{m} \sum_i |Z_i|$.

Fix i. Assume that $\{z_\beta\}$, $\beta \in \mathcal{B}_i$ is the family of the points of the grid in G_i grid in Z_i.

For some $\beta \in \mathcal{B}_i$, define $x_\beta \in X_i$ to be the point which projects onto z_β, and let H_β be the tangent hyperplane at x_β. Denote by F_β the family of points in H_β whose projection into G_i lies in $z_\beta + \frac{1}{m} W_i$.

If $\beta \in \mathcal{B}_i$ and the projection of X_i into H_β intersects F_β then $|F_\beta| \approx \frac{1}{m}$, F_β is a parallelotope, and the ratio of the circum- and the inradius of F_β is at most $1.5\sqrt{d}$. In particular, the inradius of F_β is $\gg m^{\frac{-1}{d-1}}$.

The proof is presented only for the case if $\{\Sigma_\beta\}$ covers ∂M (the proof for the other case is analogous). Then Φ_β ($\widetilde{\Phi}_\beta$) is defined by scaling F_β by a factor λ ($\tilde{\lambda}$), where

$$\lambda = 1 + \frac{1}{m^{\frac{1}{d-1}}} \quad \text{and} \quad \tilde{\lambda} = 1 + \cdot \frac{2}{m^{\frac{1}{d-1}}}.$$

Then all the conditions (i),...,(vii) are satisfied if we replace ∂M by X_i. The existence of q_β, etc. follows by the fact that Q_x, etc. is Lipschitz.

Finally, construct the subfamily of $\{\Sigma_\beta\}$ satisfying the properties (i),...,(vi) by induction on i: First, take all the patches associated to X_1. Out of the patches associated to X_{i+1}, take Σ_β if $z_\beta + \frac{1}{m} W_{i+1}$ is not contained in the projection of $(X_1 \cup \cdots \cup X_i) \cap X_{i+1}$ into G_{i+1}. The convexity of Z_i yields that

$$\left|(\partial X_i + t \cdot B^d) \cap \partial M\right| \ll t$$

holds for each i, and hence the patches we have constructed satisfy all conditions. ∎

Remark: Assume that the only condition we have is that ∂M is continuous, and let $\varepsilon > 0$. Then the same proof gives the statement above, only $m^{\frac{-1}{d-1}}$ should be replaced by ε in (i),...,(vi).

3.2 Polytopes and power diagrams.

So let Q_x be a Lipschitz function of $x \in \partial M$. For both the upper bound and the lower bound, apply Lemma 1 with $m = n^{\frac{1}{d+2}}$ and $\varrho(x) = \kappa(x)^{\frac{1}{d+1}}$ (no need for ψ). In order to simplify notation, set

$$\varepsilon = \frac{1}{n^{\frac{1}{8d^2}}}.$$

Observe that if $x \in \widetilde{\Sigma}_\beta$ and $x = (z, f_\beta(z))$ then the Gauß–Kronecker curvature at x is

$$\kappa(x) = \frac{\det q_z}{(1 + \|l_z\|^2)^{\frac{d+1}{2}}} = (1 + O(\varepsilon)) \cdot 2^{d-1} \det q_\beta. \tag{17}$$

If F is a facet at Σ_β, consider the point $(a, f_\beta(a))$ for $a \in H_\beta$ where the tangent plane is parallel to aff F. Thus aff$F$ is parameterized as

$$\varphi_\beta(z) = f_\beta(a) + l_a(z - a) + r$$

for some $r \in \mathbb{R}$. Since $f_\beta(z) = f_\beta(a) + l_a(z - a) + g_a(z - a)$ by Taylor's formula where $g_a(z - a) = \frac{1}{2} q_w(z - a)$ for some w between a and z, we have

$$f_\beta(z) - \varphi_\beta(z) = g_a(z - a) - r \quad \text{and} \quad q_\beta \le g_a \le (1 + O(\varepsilon)) \cdot q_\beta. \tag{18}$$

Therefore we can transfer the estimates about power diagrams to the patches Σ_β, and the reverse. In this section, we associate power diagrams to a well approximating polytope, and reverse. The corresponding calculations are left for the next section.

We start with the lower bound, and hence the closures of $\widetilde{\Sigma}_\beta$'s are pairwise disjoint. For large n, let P_n be the polytope with at most n vertices minimizing $\delta_S(M, P_n)$.

It is easy to construct examples of polytopes showing that $\delta_S(M, P_n) \ll n^{\frac{-2}{d-1}}$. Since the boundary of M is C_+^2, we may assume by Proposition 2.2 that if F is a facet of P_n then

$$\text{diam} F,\ \text{diam}\,(\text{aff} F \cap \partial M) \ll n^{\frac{-2}{(d-1)(d+1)}}. \tag{19}$$

Fix some β. Let φ_β be the piecewise linear function on $\widetilde{\Phi}_\beta$ such that the part of ∂P_n above $\widetilde{\Phi}_\beta$ is the graph of φ_β. Consider the family $\{(u_j, \varphi_\beta(u_j))\}$ of vertices of the graph of φ_β. There exists a power diagram $\widetilde{T}_\beta = \{\Pi_i, a_i, r_i\}$ with respect to q_β such that $\{u_j\}$ form the set of vertices of $\widetilde{T}_\beta$; namely, if $u_j \in \Pi_i$ then

$$f_\beta(u_j) - \varphi(u_j) = q_\beta(u_j - a_i) - r_i.$$

Now the associated power diagram T_β with respect to q_β is defined so that the tiles of T_β are the Π_i's which intersect Φ_β. The corresponding part Y_β of ∂P_n is the graph of φ_β above these tiles. Denote the number vertices of T_β in Φ_β by n_β.

Note that $m^{\frac{-2}{d-1}} = n^{\frac{-2}{(d-1)(d+2)}}$. We deduce by (19) and Lemma 1 that $T_\beta^+ \subset \widetilde{\Phi}_\beta$ for large n, and the distance of T_β^+ and $\partial\widetilde{\Phi}_\beta$ is $\gg n^{\frac{-2}{(d+2)(d-1)}}$.

In order to prove the upper bound we reverse the process: Define n_β so that $\sum_\beta n_\beta = n$ and n_β is proportional to $(\det q_\beta)^{\frac{1}{2}} |\Phi_\beta|$ up to $1 + O(\varepsilon)$. In particular, $n_\beta \approx n^{\frac{n+1}{n+2}}$. Now $|\widetilde{\Phi}_\beta| = (1+O(\varepsilon)) \cdot |\Phi_\beta|$ holds by Lemma 1, and hence Proposition 2.5 yields a power diagram $\widetilde{T}_\beta = \{\Pi_i, a_i, r_i\}$ covering $\widetilde{\Phi}_\beta$ such that number of vertices in $\widetilde{\Phi}_\beta$ is at most n_β and

$$v(\widetilde{T}_\beta) \le (1 + O(\varepsilon)) \cdot \text{ldel}_{d-1} \cdot (\det q_\beta)^{\frac{1}{d-1}} \cdot |\Phi_\beta|^{\frac{d+1}{d-1}} \cdot \frac{1}{n_\beta^{\frac{2}{d-1}}}.$$

Define φ_β as the piecewise linear function on $\widetilde{\Phi}_\beta$ such that the vertices of the graph of φ_β are formed by the family $\{(v, f_\beta(v))\}$ where v is some vertex of T_β, and if u is a vertex of Π_i then

$$f_\beta(u) - \varphi_\beta(u) = q_\beta(u - a_i) - r_i.$$

Define T_β to be the power diagram of the tiles Π_i which intersect Φ_β and Y_β is the graph of φ_β above these tiles.

Since the analogue of (19) is satisfied by the facets of T_β, Lemma 1 yields that $T_\beta^+ \subset \widetilde{\Phi}_\beta$, and the distance of T_β^+ and $\partial\widetilde{\Phi}_\beta$ is $\gg n^{\frac{-2}{(d+2)(d-1)}}$.

Finally, define Q_n as the convex hull of the Y_β's.

3.3 PIECING THE ESTIMATES.

First we prove a technical, but extremely useful statement.

PROPOSITION 3.1: *For any $a \in \mathbb{R}^{d-1}$, let g_a be a continuous, non-negative function. Consider $a_1, \dots, a_m \in \mathbb{R}^{d-1}$, $r_1 \dots, r_m \in \mathbb{R}$ and Jordan measurable sets $\Omega_1, \dots, \Omega_m$ such that $\Omega_1, \dots, \Omega_m$ cover the sets $g_{a_i}(z - a_i) \leq r_i$ and $g_{a_i}(z - a_i) - r_i = \min_j g_{a_j}(z - a_j) - r_j$ for $z \in \Omega_i$.*

Assume that q is a positive definite quadratic form, and $q(z-a) \leq g_a(z-a) \leq 2q(z-a)$ for every a. Then

$$\sum_i \int_{\Omega_i} g_{a_i}(z - a_i)\, dz \ll \sum_i \int_{\Omega_i} |g_{a_i}(z - a_i) - r_i|\, dz.$$

Proof: Denote by σ_0 the part of $\sigma = \bigcup \Omega_i$ which is contained in the union of the sets $g_{a_i}(z - a_i) < 2r_i$, and set $\sigma_1 = \sigma \backslash \sigma_0$. Readily,

$$\int_{\sigma_1} g_{a_i}(z - a_i)\, dz \leq 2 \cdot \int_{\sigma_1} |g_{a_i}(z - a_i) - r_i|\, dz.$$

Now number $r_1, \dots, r_m$, so that r_1 is maximal, and $g_{a_i}(z-a_i) \leq r_i$, $i = 1, \dots l$, is a maximal disjoint family with the property that if the set $g_{a_j}(z - a_j) \leq r_j$ intersects the set $g_{a_i}(z - a_i) \leq r_i$ for $j > i$ then $r_j \leq r_i$. We deduce that

$$\begin{aligned} \int_{\sigma_0} g_{a_i}(z - a_i)\, dz &\ll \sum_{i=0}^{l} \int_{q(z-a_i)<r_i} q(z - a_i)\, dz \\ &\ll \sum_{i=0}^{l} \int_{g_{a_i}(z-a_i)<r_i} |g_{a_i}(z - a_i) - r_i|\, dz \end{aligned}$$

where the last expression is readily at most $\int_{\sigma_0} |g_{a_i}(z - a_i) - r_i|\, dz$. ∎

Now we have arrived at the core of the argument; namely, that estimates can be transferred from paraboloids to "almost paraboloids".

We use the set up of the previous section, and hence T_β, n_β, and φ_β are defined as above.

PROPOSITION 3.2:

$$(1 - O(\varepsilon)) \cdot \int_{\Phi_\beta} |f_\beta(z) - \varphi_\beta(z)|\, dz \leq v(T_\beta) \leq (1 + O(\varepsilon)) \cdot \int_{\widetilde{\Phi}_\beta} |f_\beta(z) - \varphi_\beta(z)|\, dz.$$

Proof: We use the notions f_β, φ_β, T_β and q_β without index. For a vertex u of Π_i, set $\Delta(u) = q(u - a_i) - r_i$, which is in turn equal to $f(u) - \varphi(u)$ by definition.

First we claim that for any $z \in \Pi_i$,

$$|f(z) - \varphi(z)| = |q(z - a_i) - r_i| + O(\varepsilon) \max_{y \in \Pi_i} q(y - a_i). \tag{20}$$

We may assume that $f'(a_i) = 0$ and a_i is the origin in $\mathbb{R}^{d-1}$. It is sufficient to prove that

$$|\varphi(z) - r_i| = O(\varepsilon) \max_{y \in \Pi_i} q(y - a_i). \tag{21}$$

Let $u_1, \ldots, u_d$ be the vertices of Π_i and let $v_1, \ldots, v_d$ be the vertices of T_β so that $z = \sum_j s_j u_j = \sum_j t_j v_j$ and

$$\varphi(z) = \sum_j t_j (f(v_j) - \Delta(v_j))$$

with $\sum_j s_j = \sum_j t_j = 1$ and $s_j, t_j \geq 0$. Since $r_i = q(u_j) - \Delta(u_j)$ and $q(v_j) - \Delta(v_j) \geq r_i$ hold for $j = 1, \ldots, d$, we have

$$\begin{aligned} \varphi(z) &\geq \sum_j t_j \left(q(v_j) - \Delta(v_j)\right) \geq \sum_j s_j \left(q(u_j) - \Delta(u_j)\right) \quad \text{and} \\ \varphi(z) &\leq \sum_j s_j \left(f(u_j) - \Delta(u_j)\right). \end{aligned}$$

We deduce (21) by (18), and in turn (20) follows.

The inequality $|\Pi_i| \cdot \max_{z \in \Pi_i} q(z) \ll \int_\Pi q(z)\,dz$ and (20) yield that

$$\int_{\Pi_i} |f(z) - \varphi(z)|\,dz = \int_{\Pi_i} |q(z - a_i) - r_i|\,dz + O(\varepsilon) \int_{\Pi_i} q(z - a_i)\,dz.$$

Now the lower bound for $v(T)$ is a consequence of Proposition 3.1.

Turning to the upper bound for $v(T)$, denote by $\widetilde{\Pi}_i$ the set of z where $f(z) - \varphi(z) = g_{a_i}(z - a_i) - r_i$. Analogously as above, we obtain the formula

$$\int_{\widetilde{\Pi}_i} |q(z - a_i) - r_i|\,dz = \int_{\widetilde{\Pi}_i} |f(z) - \varphi(z)|\,dz + O(\varepsilon) \int_{\widetilde{\Pi}_i} g_{a_i}(z - a_i)\,dz.$$

Denote by I^* the set of indices j with $\widetilde{\Pi}_j \cap \Omega_i \neq \emptyset$ for some Ω_i corresponding to T. Then one can define the sets $\widetilde{\Omega}_j$ for $j \in I^*$ with respect to $\{\widetilde{\Pi}_j, a_j, r_j, g_{a_j}\}_{j \in I^*}$. Since $\bigcup_{j \in I^*} \{\widetilde{\Omega}_j\}$ is contained in $\widetilde{\Phi}_\beta$ for large n, we deduce the upper bound for $v(T)$, again by Proposition 3.1. ∎

First we prove the upper bound in Theorem 1. When estimating $\delta(M, Q_n)$, we have to be careful what happens when piecing. We separate the part of the boundary corresponding to boring patches; here Proposition 2.3 and Lemma 1 yield that this part causes a small error. So assume that Σ_β is not boring. Then on the part which is possibly multiple covered the integral of $|f - \varphi_\beta|$ is small (see (12) in the proof of Proposition 2.4). We deduce by Proposition 3.2 that

$$\delta(M, Q_n) \leq (1 + O(\varepsilon)) \cdot \sum_\beta v(T_\beta).$$

Here $n_\beta \approx n^{\frac{d+1}{d+2}}$, and hence Proposition 2.5 and (17) yield that

$$\begin{aligned}\delta(M, Q_n) &\leq (1 + O(\varepsilon)) \cdot \mathrm{ldel}_{d-1} \cdot \sum_\beta (\det q_\beta)^{\frac{1}{d-1}} \cdot |\Phi_\beta|^{\frac{d+1}{d-1}} \cdot \frac{1}{n_\beta^{\frac{2}{d-1}}} \\ &= (1 + O(\varepsilon)) \cdot \frac{1}{2} \mathrm{ldel}_{d-1} \cdot \sum_\beta \left(\int_{\Sigma_\beta} \kappa(x)^{\frac{1}{d+1}} dx \right)^{\frac{d+1}{d-1}} \cdot \frac{1}{n_\beta^{\frac{2}{d-1}}}.\end{aligned}$$

Finally, we conclude the upper bound in Theorem 1 by the choice of n_β.

Now we turn to the lower bound in Theorem 1. We deduce by Proposition 3.2 that

$$\delta(M, P_n) \geq (1 + O(\varepsilon)) \cdot \sum_\beta v(T_\beta).$$

Now Proposition 2.5 yields that

$$\delta(M, P_n) \geq (1 + O(\varepsilon)) \sum_{n_\beta \geq n^{\frac{d+1}{d+2}\left(1 - \frac{1}{4d}\right)}} \mathrm{ldel}_{d-1} \cdot (\det q_\beta)^{\frac{1}{d-1}} \cdot |\Phi_\beta|^{\frac{d+1}{d-1}} \cdot \frac{1}{n_\beta^{\frac{2}{d-1}}}.$$

We deduce by $\delta_S(M, P_n) \ll n^{\frac{-2}{d-1}}$ and Proposition 2.1 applied to the Φ_β's that the number of n_β with $n_\beta < n^{\frac{d+1}{d+2}(1 - \frac{1}{4d})}$ is at most $n^{\frac{1}{d+2}(1 - \frac{1}{2d})}$. Now some simple calculations and Lemma 1 yield that the area covered by Σ_β's with $n_\beta \geq n^{\frac{d+1}{d+2}(1 - \frac{1}{4d})}$ is $(1 + O(\varepsilon)) \cdot |\partial M|$. Therefore (17) and (5) yield the lower bound in Theorem 1. Therefore the proof is complete in case of general approximation if the number of vertices is bounded.

Remark: Assume that ∂M is C^2_+. Then we use the modified version of Lemma 1, as it is described in the Remark after it. Now the proof above yields the asymptotic formula (1).

3.4 Inscribed and circumscribed polytopes.

For inscribed (circumscribed) polytopes, we use inscribed (circumscribed) power diagrams. Similar arguments work as above, only one needs a little care whether piecing keeps the property being inscribed (circumscribed).

For inscribed polytopes, the same argument works word by word, as an inscribed polytope gives rise to inscribed power diagrams, and the reverse.

Let us consider the case of circumscribed polytopes. We claim that if P_n is a circumscribed polytope then each T_β is a circumscribed power diagram. To verify this statement, denote by φ^* the piecewise linear function which is associated to T_β as a power diagram with respect to q_β. It is sufficient to prove that if $\{(u_j, \varphi_\beta(u_j))\}$, $j = 1, \ldots, d$ are vertices of a facet of Y_β then $\mathrm{conv}\{(u_j, \varphi_\beta^*(u_j))\}$ is below the graph of q_β. This follows as the graph of φ_β is below the graph of f, $q_\beta(u_j) - \varphi^*(u_j) = f(u_j) - \varphi(u_j)$ and $\frac{1}{2}q_z > q_\beta$ for each z. Therefore the lower bounds during the course of the proof can be proved exactly in the same way as above.

In case of the upper bound, let $\gamma > 0$ be the constant such that $\frac{1}{2}q_z < (1 + \gamma\varepsilon)q_\beta$. The main trick is that we now define T_β with respect to $(1 + \gamma\varepsilon)q_\beta$. This causes only a multiplicative error of $1 + O(\varepsilon)$ in the estimate. Now the argument presented above shows that Y_β is circumscribed.

3.5 THE CASE OF THE FACETS.

The arguments are basically the same, actually somewhat simpler. Say in this case, we consider power diagrams with at most n tiles.

The correspondence between polytopal hypersurfaces and power diagrams is given by (18). More precisely, $\{F_i\}$, $i = 1, \ldots, n_\beta$ is the family of facets such that their projection into H_β intersects Φ_β. To each F_i and the corresponding a_i and r_i, we have that $\mathrm{aff}F_i$ is the graph of $f_\beta(a_i) + l_{a_i}(z - a_i) + r_i$. When piecing patches in the case of facets, we take intersection of the half spaces, which is equivalent to considering the union of all $\{a_i, r_i\}$ if the pieces are assigned with respect to the same quadratic form.

The proof works the analogous way for general approximation. No changes are needed in the argument when passing from general approximation to the circumscribed case (here one can actually assume that $r_i = 0$).

Next, consider the lower bound for inscribed polytopes. Now $\frac{1}{2}q_z > q_\beta$ yields that the resulting power diagram is inscribed. More precisely, if $(y, f_\beta(y))$ is below the graph of $f_\beta(a_i) + l_{a_i}(z - a_i) + r_i$ then $(y, q_\beta(y))$ is below the graph of $q_\beta(a_i) + l_{a_i}(z - a_i) + r_i$. Therefore exactly the same argument applies as in the general case.

Finally, for the lower bound, define T_β with respect to $(1 + \gamma\varepsilon)q_\beta$ where $\frac{1}{2}q_z < (1 + \gamma\varepsilon)q_\beta$, and then the resulting polytope is inscribed.

3.6 THE δ_w METRIC.

There exists a general version of the symmetric difference metric: If $w(x)$ is a positive continuous function in a neighbourhood of ∂K in $\mathbb{R}^d$, then $\delta_w(K,P)$ is the integral of $w(x)$ on the symmetric difference of K and P.

Similar arguments as for δ_S yield (see 12) that if P_n is the polytope with at most n vertices minimizing $\delta_w(M,P_n)$ then

$$\delta_w(M,P_n) \sim \frac{1}{2}\,\mathrm{ldel}_{d-1}\cdot\left(\int_{\partial M} w(x)^{\frac{d+1}{d-1}}\kappa(x)^{\frac{1}{d+1}}\,dx\right)^{\frac{d+1}{d-1}}\cdot\frac{1}{n^{\frac{2}{d-1}}}. \tag{22}$$

Now (22) can be strengthened similarly for the case of the symmetric difference metric. One only needs to assume that $w(x)$ is constant in a neighbourhood of $\widetilde{\Sigma}_\beta$ up to $1+O(\varepsilon)$ for suitable ε.

COROLLARY 1: *Assume that ∂M is C^2_+ and the second fundamental form Q_x is a Lipschitz function of x, and w is a positive Lipschitz function in an open neighbourhood of ∂M. If P_n ($P_{(n)}$) is the polytope with at most n vertices (facets) minimizing $\delta_w(M,P_n)$ then*

$$\delta_w(M,P_n) = \left(1+O\left(n^{\frac{-1}{8d^2}}\right)\right)\cdot\frac{1}{2}\,\mathrm{ldel}_{d-1}\cdot\left(\int_{\partial M} w(x)^{\frac{d+1}{d-1}}\kappa(x)^{\frac{1}{d+1}}\,dx\right)^{\frac{d+1}{d-1}}\frac{1}{n^{\frac{2}{d-1}}},$$

$$\delta_w(M,P_{(n)}) = \left(1+O\left(n^{\frac{-1}{8d^2}}\right)\right)\cdot\frac{1}{2}\,\mathrm{ldiv}_{d-1}\cdot\left(\int_{\partial M} w(x)^{\frac{d+1}{d-1}}\kappa(x)^{\frac{1}{d+1}}\,dx\right)^{\frac{d+1}{d-1}}\frac{1}{n^{\frac{2}{d-1}}}.$$

One may impose the additional condition that P_n is inscribed, or that P_n is circumscribed.

4. The L_p metric

4.1 THE L_1 METRIC.

In this case, the main tool is polarity. Since $\delta_1(M,P)$ is invariant under the translation of M and P by the same vector, we may asssume that the origin o lies in the interior of M.

Define the polar M^* of M as

$$M^* = \{x\colon \langle x,y\rangle \le 1 \quad \text{holds} \quad \forall y\in M\}.$$

Then $M^{**}=M$ and ∂M^* is also C^2_+ (see [18]).

Assume that P is a polytope containing o in its interior. Then there exists a one to one correspondence between the k-faces of P and the $(d-1-k)$-faces of P^*, and P is circumscribed around M if and only if P^* is inscribed into M.

Set $w(z) = 1/\|z\|^{\frac{1}{d+1}}$ for $z \neq o$. Now the main observation is that (see S. Glasauer and P. M. Gruber [7])

$$\delta_1(M,P) = \delta_w(M^*,P^*). \tag{23}$$

This way the problem of best approximation of X with respect to δ_1 bounding the number of k-faces is translated into best approximation of X^* with respect to δ_w bounding the number of $(d-1-k)$-faces.

Therefore (23), the formula (see S. Glasauer and P. M. Gruber [7])

$$\int_{\partial M^*} \frac{1}{\|x\|^{\frac{1}{d-1}}} \cdot \kappa_{M^*}(x)^{\frac{1}{d+1}} dx = \int_{\partial M} \kappa_M(x)^{\frac{d}{d+1}} dx$$

and Corollary 1 yield Theorem 2.

4.1 THE L_p METRIC, $p \geq 1$.

Let q be a positive definite quadratic form. What may sound as a surprise (but remember the case of the L_1 metric), we consider circumscribed power diagrams with given number of tiles. Assume that each $r_i = 0$, and hence a power diagram in $\mathbb{R}^{d-1}$ is given as $T = \{\Pi_i, a_i\}$, and $\Omega_i = \Pi_i$. For $p \geq 1$, set

$$v^p(T) = \sum_i \int_{\Omega_i} q(z-a_i)^p dz,$$

and hence $v^1(T) = v(T)$. Observe that for $\lambda > 0$,

$$v^p(\lambda T) = \lambda^{d-1+2p} \cdot v^p(T).$$

If C is a convex body in $\mathbb{R}^{d-1}$ then define

$$v^p_{(n)}(C,q) = \min\{v^p(T)\colon T \text{ covers } C \text{ and has at most } n \text{ tiles}\}.$$

Note that if $T = \{\Pi_i, a_i\}$ is a power diagram with at most n tiles and covering C, and $v^p(T) < 2v^p_{(n)}(C,q)$, then the analogue of Proposition 2.2 yields for any Π_i that

$$\operatorname{diam}\Pi \ll n^{\frac{-2p}{(d-1)(d-1+2p)}}. \tag{24}$$

Then analogously as for $v_n(P,q)$, we deduce

PROPOSITION 4.1: *In $\mathbb{R}^{d-1}$, let P be a parallelotope and let q be a positive definite quadratic form. If the ratio of the circum- and the inradius with respect to q is at most $2\sqrt{d}$ then*

$$v^p_n(P,q) = \left(1 + O\left(n^{\frac{-p}{10d(d+p)}}\right)\right) \cdot \tilde{c}_{p,d} \cdot (\det q)^{\frac{p}{d-1}} \cdot |C|^{\frac{d-1+2p}{d-1}} \cdot \frac{1}{n^{\frac{2}{d-1}}}$$

where $c_{p,d}$ depends on p and d, and $O(\cdot)$ depends on M, p and d

How does it connect to polytopal approximation? Denote by $\nu(x)$ the exterior unit normal at $x \in \partial M$. If $P \subset M$ is a polytope then

$$\delta_p(M,P) = \left(\int_{\partial M} \min_{v \text{ vetrex}} \langle \nu(x), x - v \rangle^p \cdot \kappa(x)\, dx \right)^{\frac{1}{p}}. \tag{25}$$

So let $P_n \subset M$ be a polytope with at most n vertices and minimizing $\delta_p(M,P_n)$. Readily, all the vertices are contained in ∂M. First we describe how to get the lower bound.

It is easy to show that $\delta_p(M,P_n) \ll n^{\frac{-2}{d-1}}$. Since the boundary of M is C^2_+, the analogue of Proposition 2.2 yields that if F is a facet of such a P then

$$\operatorname{diam} F \ll n^{\frac{-2p}{(d-1)(d-1+2p)}}. \tag{26}$$

So apply Lemma 1 with $m = n^{\frac{p}{d+2p}}$, and constructing disjoint Σ_β's. Set $\varepsilon = n^{\frac{-p}{20d(d+2p)}}$.

For given β, define $\widetilde{T}_\beta = \{\Pi_j, a_j\}$ to be the power diagram with respect to q_β, such that $\{(a_j, f_\beta(a_j))\}$ is the the family of vertices of P_n with distance at most $n^{\frac{-2p}{(d-1)(d+2p)}}$ from $\widetilde{\Sigma}_\beta$. Denote by n_β the number of the tiles of $\widetilde{T}_\beta$.

Observe that if $x \in \widetilde{\Sigma}_\beta$ and the vertex v minimizes $\langle \nu(x), x - v\rangle$ among the vertices of P_n then $d(x,v) < n^{\frac{-2p}{(d-1)(d+2p)}}$. On the other hand, if $d(x,v_i) < n^{\frac{-2p}{(d-1)(d+2p)}}$ for a vertex v_i and $x = (z, f_\beta(z))$ then

$$\langle \nu(x), x - v_i \rangle = (1 + O(\varepsilon)) \cdot q_\beta(z - a_i).$$

Therefore

$$\delta_p(M,P) \geq (1 + O(\varepsilon)) \cdot \left(\sum_\beta 2^{d-1} \det q_\beta \cdot v(T_\beta) \right)^{\frac{1}{p}}.$$

Now the proof can be finished as in the case of the symmetric difference metric. Note that $c_{p,d} = \frac{1}{2}\tilde{c}_{p,d}$.

The upper bound in Theorem 3 can be verified using the similar alterations of the earlier argument as above.

5. General convex C^2_+ hypersurfaces

We say that X is a convex C^2_+ hypersurface if it is an open, Jordan measurable subset of a convex body M, the origin lies in the interior of M, and the closure of X is contained in an open, C^2_+ subset of ∂M.

Similarly, Y is called a convex polytopal hypersurface if it is a Jordan measurable subset of a polytope P and the origin lies in the interior of P. If Y approximates X then we make the following assumptions: If the approximation is with respect to the symmetric difference metric (or δ_w) then define $Y \subset \partial P$ as the radial projection of X. Otherwise, for any $x \in X$ consider the points $y \in \partial P$ where the exterior normals at x to M are also exterior normals at y to P, and Y is the union of these sets. We say that Y is inscribed if $Y \subset M$, and Y is circumscribed if $Y \cap \mathrm{int} M = \emptyset$. The faces of Y are the intersections of the faces of P with the interior of Y.

Now we extend the notions of distances to X and Y. Observe that for $x \in X$, we have

$$h_P(\nu(x)) - h_M(\nu(x)) = \max_{y\in Y}\langle \nu(x), y - x\rangle.$$

SYMMETRIC DIFFERENCE METRIC AND δ_w: $\delta_S(X,Y)$ is the volume of the part of the cone over X which lies between X and Y, and $\delta_w(X,Y)$ is the integral of w on this part.

L_p METRIC, $p \geq 1$: $\delta_p(X,Y) = \left(\int_X |\max_{y\in Y}\langle \nu(x), y-x\rangle|^p \,\kappa(x)\,dx\right)^{\frac{1}{p}}$.

Observe that if $X = \partial M$ (and hence $Y = \partial P$) then $\delta(X,Y) = \delta(M,P)$.

In case of the L_1 metric, we have a closer look at the properties of polarity. If $u \neq o$ then define u^* to be the hyperplane $H = \{z: \langle z, u\rangle = 1\}$, and set $H^* = u$. Observe that if $v \in u^*$ then $u \in v^*$.

Let X be a convex C^2_+ hypersurface, which then lies on the boundary of a convex body M where M contains the origin in its interior. Define X^* to be the set of polar images of the tangent hyperplanes at the points of X. Then X^* is also convex C^2_+ hypersurface lying on the boundary of M^* (see [18]). Observe that $X^{**} = X$.

Let $Y \subset \partial P$ be a convex polytopal surface approximating X with respect to δ_1. Consider the tangent hyperplanes at the points of Y which are parallel to the tangent hyperplane at some point of X, and denote by Y^* the set of polar images of them. Then $Y^* \subset \partial P^*$ is a convex polytopal hypersurface approximating X^* in the sense of δ_w.

Now there exists a one to one correspondence between the k-faces of Y whose closure does not intersect the boundary of Y and the $(d-1-k)$-faces of Y^*.

Set $w(x) = \|x\|^{-(d+1)}$ for $x \neq o$. Then the same argument as above yields that

$$\delta_1(X,Y) = \delta_w(X^*,Y^*).$$

This way the problem of best approximation of X with respect to δ_1 bounding the number of k-faces is translated into best approximation of X^* with respect to δ_w bounding the number of $(d-1-k)$-faces.

Call a set rectifiable if it is the finite union of images of compact Jordan measurable subsets of $\mathbb{R}^{d-2}$ by Lipschitz maps. Note that if σ is a rectifiable subset of $\mathbb{R}^{d-1}$ then for small t (see H. Federer [5], but rather prove for yourself),

$$|\sigma + tB^{d-1}| \ll_\sigma t. \tag{27}$$

Therefore, if the boundary of X is rectifiable then it causes a smaller error than the error we accumulate otherwise. In particular, the method above for closed convex hypersurfaces yields

COROLLARY 2: *Let X be C_+^2 such that the second fundamental form Q_x is a Lipschitz function of x and the boundary of X is rectifiable. Assume that Y_n is a best approximating surface with respect to the metric δ having at most n vertices.*

(i) *If $\delta = \delta_S$ then*

$$\delta_S(X, Y_n) = \left(1 + O\left(n^{\frac{-1}{8d^2}}\right)\right) \cdot \frac{1}{2}\,\mathrm{ldel}_{d-1} \cdot \left(\int_X \kappa(x)^{\frac{1}{d+1}}\,dx\right)^{\frac{d+1}{d-1}} \cdot \frac{1}{n^{\frac{2}{d-1}}}.$$

The analogous formula holds if Y_n is assumed to be inscribed or circumscribed, or the number of facets is bounded.

(ii) *If $\delta = \delta_1$ then*

$$\delta_1(X, Y_n) = \left(1 + O\left(n^{\frac{-1}{8d^2}}\right)\right) \cdot \frac{1}{2}\,\mathrm{ldiv}_{d-1} \cdot \left(\int_X \kappa(x)^{\frac{d}{d+1}}\,dx\right)^{\frac{d+1}{d-1}} \cdot \frac{1}{n^{\frac{2}{d-1}}}.$$

The analogous formula holds if Y_n is assumed to be inscribed or circumscribed, or the number of facets is bounded.

(iii) *If $p > 1$, $\delta = \delta_p$ and Y_n is inscribed then*

$$\delta_p(X, Y_n) = \left(1 + O\left(n^{\frac{-p}{20d(d+p)}}\right)\right) \cdot c_{p,d} \cdot \left(\int_X \kappa(x)^{\frac{d-1+p}{d-1+2p}}\,dx\right)^{\frac{d-1+2p}{p(d-1)}} \cdot \frac{1}{n^{\frac{2}{d-1}}}.$$

Finally, let us formulate the geometric version of Proposition 2.3. This statement can be useful when piecing patches in later applications. The proof is again based on Proposition 2.3, and on subdividing X into almost paraboloid patches.

PROPOSITION 5.1: *Assume that ∂M is C_+^2, and X is an open, Jordan measurable subset of ∂M. Then there exists a $\Delta > 0$ such that if $Y_1, \ldots, Y_m$ are polytopal hypersurfaces approximating Σ and the facets of the Y_i's have diameter at most*

Δ then the polytopal hypersurface Y determined by the vertices of $Y_1, \ldots, Y_m$ satisfies

$$\delta_S(\Sigma, Y) \ll \delta_S(Y_1) + \cdots + \delta_S(Y_m).$$

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